

# Galilean geometry of motions

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## Abstract

In this paper we show that Galilean group is a matrix Lie group and find its structure. Then provide the invariants of special Galilean geometry of motions, by Olver's method of moving coframes, we also find the corresponding  $\{e\}$ -structure.

Key words: Equivalence of sub-manifolds, Moving coframe, Galilean space.

A.M.S. 2000 Subject Classification: 58D19, 70E15.

## 1 Introduction:

The method of moving coframes is one of the cornerstones of our investigations for finding the invariants of the equivalence problem which is done in [1] and [2]. Let us summarize the basic method of moving coframes.

The basic steps are:

- (i) Determine the moving frame of order zero, by choosing a base point and solving for Galilean group action.
- (ii) Determine the invariant forms in this case (the finite dimensional), they are the Maurer-Cartan forms, which computed by direct use of the transformation group formulae but not the matrix approach.
- (iii) Use the invariant lift to pull-back the invariant forms, leading to the moving coframe of order zero.
- (iv) Determine the lifted invariants by finding the linear dependencies among the restricted to horizontal components of the moving coframe forms.
- (v) Normalize any group-dependent invariants to convenient constant values by solving for some of the unspecified parameters.
- (vi) Successively eliminate parameters by substituting the normalization formulae into the moving coframe and recomputing dependencies.

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- (vii) After the parameters have all been normalized, the differential invariants will appear through any remaining dependencies among the final moving coframe elements. The Invariant differential operators are found as the dual differential operators to a basis for the invariant coframe forms.

At first we define some basic prerequisite from Galilean group. Explanatory details are found in [3] , [4].

**Definition 1.** The Galilean group is defined as

$$\text{Gal}(3) = \left\{ \begin{bmatrix} 1 & 0 & s \\ \mathbf{v} & R & \mathbf{y} \\ 0 & 0 & 1 \end{bmatrix} \middle| s \in \mathbb{R}, \mathbf{y}, \mathbf{v} \in \mathbb{R}^3, R \in \text{O}(3) \right\}$$

with a natural closed Lie subgroup structure of  $\text{GL}(5; \mathbb{R})$  as

$$\begin{aligned} \begin{bmatrix} 1 & 0 & s_1 \\ \mathbf{v}_1 & R_1 & \mathbf{y}_1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & s_2 \\ \mathbf{v}_2 & R_2 & \mathbf{y}_2 \\ 0 & 0 & 1 \end{bmatrix} &= \\ &= \begin{bmatrix} 1 & 0 & s_1 + s_2 \\ \mathbf{v}_1 + R_1 \mathbf{v}_2 & R_1 R_2 & \mathbf{y}_1 + s_2 \mathbf{v}_1 + R_1 \mathbf{y}_2 \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & s \\ \mathbf{v} & R & \mathbf{y} \\ 0 & 0 & 1 \end{bmatrix}^{-1} &= \begin{bmatrix} 1 & 0 & -s \\ -R^{-1} \mathbf{v} & R^{-1} & R^{-1}(s \mathbf{v} - \mathbf{y}) \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

This is a 10-dimensional Lie group. The *special Galilean group* is defined as connected component of  $e$  in  $\text{Gal}(3)$  and denoted by  $\text{SGal}(3)$ .

**Definition 2.** Let we identify the  $\mathbb{R}^4$  by

$$\mathbb{R}^4 = \left\{ \begin{bmatrix} t \\ \mathbf{x} \\ 1 \end{bmatrix} \middle| t \in \mathbb{R}, \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \right\}$$

with the natural 4-manifold structure. Then, we can define naturally the smooth action of  $\text{Gal}(3)$  on  $\mathbb{R}^4$  as

$$\begin{bmatrix} 1 & 0 & s \\ \mathbf{v} & R & \mathbf{y} \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} t \\ \mathbf{x} \\ 1 \end{bmatrix} = \begin{bmatrix} t + s \\ R \mathbf{x} + t \mathbf{v} + \mathbf{y} \\ 1 \end{bmatrix}$$

By elementary algebraic computations, we find the structure of special Galilean group,

**Theorem 1.** *Let*

$$G_1 = \left\{ \left[ \begin{array}{ccc} 1 & 0 & 0 \\ \mathbf{v} & R & \mathbf{0} \\ 0 & 0 & 1 \end{array} \right] \middle| \mathbf{v} \in \mathbb{R}^3, R \in \text{O}(3) \right\} \leq \text{GL}(4; \mathbb{R})$$

*be the group of uniformly special Galilean motions,*

$$G_2 = \left\{ \left[ \begin{array}{ccc} 1 & 0 & s \\ \mathbf{0} & I_3 & \mathbf{y} \\ 0 & 0 & 1 \end{array} \right] \middle| s \in \mathbb{R}, \mathbf{y} \in \mathbb{R}^3 \right\} \cong (\mathbb{R}^4, +)$$

*be the group of shifts of origin,*

$$G_3 = \left\{ \left[ \begin{array}{ccc} 1 & 0 & 0 \\ \mathbf{0} & R & \mathbf{0} \\ 0 & 0 & 1 \end{array} \right] \middle| R \in \text{SO}(3) \right\} \cong \text{SO}(3)$$

*be the group of rotations of reference frame, and*

$$G_4 = \left\{ \left[ \begin{array}{ccc} 1 & 0 & 0 \\ \mathbf{v} & I_3 & \mathbf{0} \\ 0 & 0 & 1 \end{array} \right] \middle| \mathbf{v} \in \mathbb{R}^3 \right\} \cong (\mathbb{R}^3, +)$$

*be the group of uniformly frame motions. Then,  $G_2 \trianglelefteq \text{SGal}(3)$ ,  $\text{SGal}(3) \cong G_1 \rtimes G_2$ ,  $G_4 \trianglelefteq G_1$ ,  $G_1 \cong G_3 \rtimes G_4$ , and  $\text{SGal}(3) \cong (\text{SO}(3) \rtimes \mathbb{R}^3) \rtimes \mathbb{R}^4$ .*

In the following theorem, we explain the algebraic structure of the infinitesimal group action  $\widehat{\mathfrak{Gal}}(3)$  induced by the action  $\text{SGal}(3)$  on  $\mathbb{R}^4$ ,

**Theorem 2.** *The Lie algebra of infinitesimal group action  $\widehat{\mathfrak{Gal}}(3) = \text{Span}_{\mathbb{R}}\{\widehat{X}_1, \dots, \widehat{X}_{10}\}$  induced by the action  $\text{SGal}(3)$  on  $\mathbb{R}^4$ , has infinitesimal generators:*

$$\begin{array}{lll} \widehat{X}_2 = \partial_x, & \widehat{X}_5 = t \partial_x, & \widehat{X}_8 = y \partial_x - x \partial_y, \\ \widehat{X}_1 = \partial_t, & \widehat{X}_3 = \partial_y, & \widehat{X}_6 = t \partial_y, & \widehat{X}_9 = x \partial_z - z \partial_x, \\ & \widehat{X}_4 = \partial_z, & \widehat{X}_7 = t \partial_z, & \widehat{X}_{10} = z \partial_y - y \partial_z, \end{array}$$

with the following structure:

	$\hat{X}_1$	$\hat{X}_2$	$\hat{X}_3$	$\hat{X}_4$	$\hat{X}_5$	$\hat{X}_6$	$\hat{X}_7$	$\hat{X}_8$	$\hat{X}_9$	$\hat{X}_{10}$
$\hat{X}_1$	0	0	0	0	$\hat{X}_2$	$\hat{X}_3$	$\hat{X}_4$	0	0	0
$\hat{X}_2$	0	0	0	0	0	0	0	$-\hat{X}_3$	$\hat{X}_4$	0
$\hat{X}_3$	0	0	0	0	0	0	0	$\hat{X}_2$	0	$-\hat{X}_4$
$\hat{X}_4$	0	0	0	0	0	0	0	0	$-\hat{X}_2$	$\hat{X}_3$
$\hat{X}_5$	$-\hat{X}_2$	0	0	0	0	0	0	$-\hat{X}_6$	$\hat{X}_7$	0
$\hat{X}_6$	$-\hat{X}_3$	0	0	0	0	0	0	$\hat{X}_5$	0	$-\hat{X}_7$
$\hat{X}_7$	$-\hat{X}_4$	0	0	0	0	0	0	0	$-\hat{X}_5$	$\hat{X}_6$
$\hat{X}_8$	0	$\hat{X}_3$	$-\hat{X}_2$	0	$\hat{X}_6$	$-\hat{X}_5$	0	0	$-\hat{X}_{10}$	$\hat{X}_9$
$\hat{X}_9$	0	$-\hat{X}_4$	0	$\hat{X}_2$	$-\hat{X}_7$	0	$\hat{X}_5$	$\hat{X}_{10}$	0	$-\hat{X}_8$
$\hat{X}_{10}$	0	0	$\hat{X}_4$	$-\hat{X}_3$	0	$\hat{X}_7$	$-\hat{X}_6$	$-\hat{X}_9$	$\hat{X}_8$	0

## 2 Computation of Maurer-Cartan forms

**Definition 3.** By multiplying 3 rotations

$$\begin{bmatrix} \cos \theta_3 & \sin \theta_3 & 0 \\ -\sin \theta_3 & \cos \theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \cos \theta_2 & 0 & \sin \theta_2 \\ 0 & 1 & 0 \\ -\sin \theta_2 & 0 & \cos \theta_2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_1 & \sin \theta_1 \\ 0 & -\sin \theta_1 & \cos \theta_1 \end{bmatrix}$$

respectively about  $x$ ,  $y$  and  $z$ -axis, we define

$$R = \begin{bmatrix} \cos \theta_2 \cos \theta_3 & \cos \theta_1 \sin \theta_3 - \sin \theta_1 \sin \theta_2 \cos \theta_3 & \sin \theta_1 \sin \theta_3 + \cos \theta_1 \sin \theta_2 \cos \theta_3 \\ -\cos \theta_2 \sin \theta_3 & \cos \theta_1 \cos \theta_3 + \sin \theta_1 \sin \theta_2 \sin \theta_3 & \sin \theta_1 \cos \theta_3 - \cos \theta_1 \sin \theta_2 \sin \theta_3 \\ -\sin \theta_2 & -\sin \theta_1 \cos \theta_2 & \cos \theta_1 \cos \theta_2 \end{bmatrix}.$$

which is an arbitrary element of  $\text{SO}(3)$ .

In order to determine the Maurer-Cartan forms, we would rather use the direct method, more details found in page 10 of [1].

**Theorem 3.** *The independent Maurer-Cartan 1-forms of  $\text{SGal}(3)$  are*

$$\begin{aligned} \mu_1 &= ds, \\ \mu_2 &= \sin \theta_2 d\theta_1 - d\theta_3, \\ \mu_3 &= \cos \theta_2 \cos \theta_3 d\theta_1 - \sin \theta_3 d\theta_2, \end{aligned}$$

$$\begin{aligned}
\mu_4 &= \cos \theta_2 \sin \theta_3 d\theta_1 + \cos \theta_3 d\theta_2, \\
\mu_5 &= \cos \theta_1 \cos \theta_2 dv_1 - \sin \theta_1 \cos \theta_2 dv_2 - \sin \theta_2 dv_3, \\
\mu_6 &= \left( \cos \theta_1 \cos \theta_2 v_1 - \sin \theta_1 \cos \theta_2 v_2 - \sin \theta_2 v_3 \right) ds \\
&\quad - \cos \theta_1 \cos \theta_2 dy_1 + \sin \theta_1 \cos \theta_2 dy_2, \\
\mu_7 &= -\left( \sin \theta_1 \sin \theta_3 + \cos \theta_1 \sin \theta_2 \cos \theta_3 \right) dv_1 \\
&\quad + \left( \sin \theta_1 \sin \theta_2 \cos \theta_3 - \cos \theta_1 \sin \theta_3 \right) dv_2 - \cos \theta_2 \cos \theta_3 dv_3, \\
\mu_8 &= \left( \cos \theta_1 \sin \theta_2 \sin \theta_3 - \sin \theta_1 \cos \theta_3 \right) dv_1 \\
&\quad - \left( \sin \theta_1 \sin \theta_2 \sin \theta_3 + \cos \theta_1 \cos \theta_3 \right) dv_2 + \cos \theta_2 \sin \theta_3 dv_3, \\
\mu_9 &= \left( (\sin \theta_1 \sin \theta_3 + \cos \theta_1 \sin \theta_2 \cos \theta_3) v_1 + (\cos \theta_1 \sin \theta_3 \right. \\
&\quad \left. - \sin \theta_1 \sin \theta_2 \cos \theta_3) v_2 + \cos \theta_2 \cos \theta_3 v_3 \right) ds \\
&\quad - \left( \cos \theta_1 \sin \theta_2 \cos \theta_3 + \sin \theta_1 \sin \theta_3 \right) dy_1 \\
&\quad + \left( -\cos \theta_1 \sin \theta_3 + \sin \theta_1 \sin \theta_2 \cos \theta_3 \right) dy_2 - \cos \theta_2 \cos \theta_3 dy_3, \\
\mu_{10} &= \left( (\sin \theta_1 \cos \theta_3 - \cos \theta_1 \sin \theta_2 \sin \theta_3) v_1 + (\cos \theta_1 \cos \theta_3 \right. \\
&\quad \left. - \sin \theta_1 \sin \theta_2 \sin \theta_3) v_2 + \cos \theta_2 \sin \theta_3 v_3 \right) ds \\
&\quad - \left( \cos \theta_1 \sin \theta_2 \cos \theta_3 + \sin \theta_1 \sin \theta_3 \right) dy_1 \\
&\quad + \left( \sin \theta_1 \sin \theta_2 \cos \theta_3 - \cos \theta_1 \sin \theta_3 \right) dy_2 - \cos \theta_2 \cos \theta_3 dy_3.
\end{aligned}$$

*Proof:* Given  $g \in \text{SGal}(3)$  and  $\mathbf{z} \in \mathbb{R}^4$ , we explicitly write the group transformation  $\bar{\mathbf{z}} = g \cdot \mathbf{z}$  in coordinate form:

$$\begin{aligned}
\bar{z}_1 &= H^1(\mathbf{z}, g) = t + s \\
\bar{z}_2 &= H^2(\mathbf{z}, g) \\
&= v_1 t + (\cos \theta_2 \cos \theta_1) x_1 + (\cos \theta_3 \sin \theta_1 - \sin \theta_3 \sin \theta_2 \cos \theta_1) x_2 \\
&\quad + (\sin \theta_3 \sin \theta_1 + \cos \theta_3 \sin \theta_2 \cos \theta_1) x_3 + y_1 \\
\bar{z}_3 &= H^3(\mathbf{z}, g) \\
&= v_2 t - (\cos \theta_2 \sin \theta_1) x_1 + (\cos \theta_3 \cos \theta_1 + \sin \theta_3 \sin \theta_2 \sin \theta_1) x_2 \\
&\quad + (\sin \theta_3 \cos \theta_1 - \cos \theta_3 \sin \theta_2 \sin \theta_1) x_3 + y_2 \\
\bar{z}_4 &= H^4(\mathbf{z}, g) \\
&= v_3 t - \sin \theta_2 x_1 - (\sin \theta_3 \cos \theta_2) x_2 + (\cos \theta_3 \cos \theta_2) x_3 + y_3.
\end{aligned}$$

We then compute the differentials of the group transformations:

$$d\bar{z}_i = \sum_{k=1}^4 \frac{\partial H^i}{\partial z_k} dz_k + \sum_{j=1}^{10} \frac{\partial H^i}{\partial g^j} dg^j, \quad i = 1, \dots, 4,$$

or more compactly

$$d\bar{\mathbf{z}} = H_{\mathbf{z}} d\mathbf{z} + H_g dg. \quad (1)$$

Next, set  $d\bar{\mathbf{z}} = 0$  in (1), and solve the resulting system of linear equations for the differentials  $dz_k$ . This leads to the formulae

$$-d\mathbf{z} = F dg = (H_{\mathbf{z}}^{-1} \cdot H_g) dg,$$

or, in full detail,

$$-dz_k = \sum_{j=1}^{10} F_j^k(\mathbf{z}, g) dg^j, \quad i = 1, \dots, 4. \quad (2)$$

Then, for each  $k$  and each fixed  $\mathbf{z}_0 \in \mathbb{R}^4$ , the one-form  $\mu_0 = \sum_{j=1}^{10} F_j^k(\mathbf{z}_0, g) dg^j$  is a left-invariant Maurer-Cartan form on the group  $\text{SGal}(3)$ . Alternatively, if one expands the right hand side of (2) in power series in  $\mathbf{z}$ ,

$$\sum_{j=1}^{10} F_j^k(\mathbf{z}, g) dg^j = \sum_{i=0}^{\infty} z_i \mu_i,$$

then each coefficient  $\mu_i$  also forms a left-invariant Maurer-Cartan form on  $\text{SGal}(3)$ .  $\square$

### 3 Zero order moving coframes

Throughout this paper, we remind you that  $\text{SGal}(3)$  is a 10-dimensional Lie group and  $M$  is a 4-dimensional manifold.

**Definition 4.** A smooth map  $\rho^{(0)} : M \rightarrow \text{SGal}(3)$  is called a compatible lift with base point  $z_0$  if it satisfies

$$\rho^{(0)}(z).z_0 = z, \quad z \in M.$$

Now, let  $\rho^{(0)} : M \rightarrow \text{SGal}(3)$  be a compatible lift with base point  $\mathbf{z}_0 = [0, \mathbf{0}, 1]^T \in \mathbb{R}^4$ , then we have,  $s = t$  and  $\mathbf{y} = \mathbf{x}$ . Thus,

**Theorem 4.** *The most general zero order compatible lift has the form*

$$\rho^{(0)}(t, \mathbf{x}; \mathbf{v}, \theta) = \begin{bmatrix} 1 & 0 & t \\ \mathbf{v} & R & \mathbf{x} \\ 0 & 0 & 1 \end{bmatrix}.$$

The next step is to characterize the group transformations by a collection of differential forms. In the finite-dimensional situation that we are currently considering, these will be obtained by pulling back the left-invariant Maurer-Cartan forms  $\mu$  on  $\text{SGal}(3)$  to the order zero moving frame bundle  $\mathcal{B}_0$  using the compatible lift.

The resulting one-forms  $\zeta^{(0)} = \rho^{(0)*}\mu$  will provide an invariant coframe on  $\mathcal{B}_0$ , which we name the *moving coframe of the zero order*. The moving coframe forms  $\zeta^{(0)}$  clearly satisfy the same Maurer-Cartan structure equations. Thus

**Theorem 5.** *The zero order moving coframe is*

$$\begin{aligned}
\zeta_1^{(0)} &= dt, \\
\zeta_2^{(0)} &= \sin \theta_2 d\theta_1 - d\theta_3, \\
\zeta_3^{(0)} &= \cos \theta_2 \cos \theta_3 d\theta_1 - \sin \theta_3 d\theta_2, \\
\zeta_4^{(0)} &= \cos \theta_2 \sin \theta_3 d\theta_1 + \cos \theta_3 d\theta_2, \\
\zeta_5^{(0)} &= \cos \theta_1 \cos \theta_2 dv_1 - \sin \theta_1 \cos \theta_2 dv_2 - \sin \theta_2 dv_3, \\
\zeta_6^{(0)} &= \left( \cos \theta_1 \cos \theta_2 v_1 - \sin \theta_1 \cos \theta_2 v_2 - \sin \theta_2 v_3 \right) dt \\
&\quad - \cos \theta_1 \cos \theta_2 dx_1 + \sin \theta_1 \cos \theta_2 dx_2, \\
\zeta_7^{(0)} &= - \left( \sin \theta_1 \sin \theta_3 + \cos \theta_1 \sin \theta_2 \cos \theta_3 \right) dv_1 \\
&\quad + \left( \sin \theta_1 \sin \theta_2 \cos \theta_3 - \cos \theta_1 \sin \theta_3 \right) dv_2 - \cos \theta_2 \cos \theta_3 dv_3, \\
\zeta_8^{(0)} &= \left( \cos \theta_1 \sin \theta_2 \sin \theta_3 - \sin \theta_1 \cos \theta_3 \right) dv_1, \\
&\quad - \left( \sin \theta_1 \sin \theta_2 \sin \theta_3 + \cos \theta_1 \cos \theta_3 \right) dv_2 + \cos \theta_2 \sin \theta_3 dv_3, \\
\zeta_9^{(0)} &= \left( (\sin \theta_1 \sin \theta_3 + \cos \theta_1 \sin \theta_2 \cos \theta_3) v_1 + (\cos \theta_1 \sin \theta_3 \right. \\
&\quad \left. - \sin \theta_1 \sin \theta_2 \cos \theta_3) v_2 + \cos \theta_2 \cos \theta_3 v_3 \right) dt \\
&\quad - \left( \cos \theta_1 \sin \theta_2 \cos \theta_3 + \sin \theta_1 \sin \theta_3 \right) dx_1 \\
&\quad + \left( -\cos \theta_1 \sin \theta_3 + \sin \theta_1 \sin \theta_2 \cos \theta_3 \right) dx_2 - \cos \theta_2 \cos \theta_3 dx_3, \\
\zeta_{10}^{(0)} &= \left( (\sin \theta_1 \cos \theta_3 - \cos \theta_1 \sin \theta_2 \sin \theta_3) v_1 + (\cos \theta_1 \cos \theta_3 \right. \\
&\quad \left. - \sin \theta_1 \sin \theta_2 \sin \theta_3) v_2 + \cos \theta_2 \sin \theta_3 v_3 \right) dt \\
&\quad - \left( \cos \theta_1 \sin \theta_2 \cos \theta_3 + \sin \theta_1 \sin \theta_3 \right) dx_1 \\
&\quad + \left( \sin \theta_1 \sin \theta_2 \cos \theta_3 - \cos \theta_1 \sin \theta_3 \right) dx_2 - \cos \theta_2 \cos \theta_3 dx_3,
\end{aligned}$$

which forms a basis for the space of one-forms on  $\mathcal{B}_0 = \mathbb{R}^4 \times G_1$   $\square$ .

## 4 First order moving coframes

**Definition 5.** A motion is a curve coincides with the graph of a function  $\mathbf{x} = \mathbf{x}(t) : \mathbb{R} \rightarrow \mathbb{R}^3$ .

We restrict the moving coframe forms to the motion (curve), which amounts to replacing the differential  $dx$  by its "horizontal" component  $\mathbf{x}_t dt$ . If we interpret the derivative  $\mathbf{x}_t$  as a coordinate on the first jet space  $J^1 = J^1(\mathbb{R}^1; \mathbb{R}^3) \cong \mathbb{R}^7$  of motions in  $\mathbb{R}^4$ , then the restriction of a differential form to the motion can be reinterpreted as the natural projection of the one-form  $dx$  on  $J^1$  to its horizontal component, using the canonical decomposition of differential forms on the jet space into horizontal and contact components. Indeed, the vertical component of the

form  $dx$  is the contact form  $d\mathbf{x} - \mathbf{x}_t dt$ , which vanishes on all prolonged sections of the first jet bundle  $J^1(\mathbb{R}^1; \mathbb{R}^3)$ . Therefore,

**Theorem 6.** *The restricted (or horizontal) moving coframe forms are defined on 7-dimensional manifold  $\{J^1\mathbf{x} = (t, \mathbf{x}(t), \mathbf{x}_t(t))\} \times G_1 \subset J^1\mathcal{B}_0$  and explicitly given by  $\eta_i^{(0)} = \zeta_i^{(0)}$ , for  $i = 1, 2, 3, 4, 5, 7, 8$ , and their linear dependencies are  $\eta_6^{(0)} = j_1 \eta_1^{(0)}$ ,  $\eta_9^{(0)} = j_2 \eta_1^{(0)}$  and  $\eta_{10}^{(0)} = j_3 \eta_1^{(0)}$ , where*

$$\begin{aligned} J_1 &= -\cos \theta_1 \cos \theta_2 v_1 + \sin \theta_1 \cos \theta_2 v_2 + \sin \theta_2 v_3 \\ &\quad + \cos \theta_1 \cos \theta_2 x'_1 + \sin \theta_1 \cos \theta_2 x'_2 - \sin \theta_2 x'_3, \\ J_2 &= (\cos \theta_1 \sin \theta_2 \sin \theta_3 - \sin \theta_1 \cos \theta_3) v_1 \\ &\quad - (\cos \theta_1 \cos \theta_3 + \sin \theta_1 \sin \theta_2 \sin \theta_3) v_2 + \cos \theta_2 \sin \theta_3 v_3 \\ &\quad + (\sin \theta_1 \cos \theta_3 - \cos \theta_1 \sin \theta_2 \sin \theta_3) x'_1 \\ &\quad + (\sin \theta_1 \sin \theta_2 \sin \theta_3 + \cos \theta_1 \cos \theta_3) x'_2 - \cos \theta_2 \sin \theta_3 x'_3, \\ J_3 &= -(\sin \theta_1 \sin \theta_3 + \cos \theta_1 \sin \theta_2 \cos \theta_3) v_1 \\ &\quad + (\sin \theta_1 \sin \theta_2 \cos \theta_3 - \cos \theta_1 \sin \theta_3) v_2 - \cos \theta_2 \cos \theta_3 v_3 \\ &\quad + (\sin \theta_1 \sin \theta_3 + \cos \theta_1 \sin \theta_2 \cos \theta_3) x'_1 \\ &\quad + (\cos \theta_1 \sin \theta_3 - \sin \theta_1 \sin \theta_2 \cos \theta_3) x'_2 + \cos \theta_2 \cos \theta_3 x'_3. \end{aligned}$$

By assumptions  $J_1 = J_2 = J_3 = 0$ , we have  $\mathbf{v} = \mathbf{x}_t$ . Thus,

**Theorem 7.** *The first order compatible lift has the form:*

$$\rho^{(1)}(t, \mathbf{x}; \mathbf{x}_t, \theta) = \begin{bmatrix} 1 & 0 & t \\ \mathbf{x}_t & R & \mathbf{x} \\ 0 & 0 & 1 \end{bmatrix}.$$

The resulting one-forms  $\zeta^{(1)} = \rho^{(1)*}\mu$  will provide an invariant coframe on  $\mathcal{B}_1$ , which we name the *moving coframe of the first order*. By substituting the map  $\rho^{(1)}$  in  $\zeta^{(0)}$  and restricting to the first prolongation or jet of the motion, namely  $\mathbf{x} = \mathbf{x}(t)$ ,  $\mathbf{x}_t = \mathbf{x}'(t)$  we have,

**Theorem 8.** *The first order moving coframe is*

$$\begin{aligned} \zeta_1^{(1)} &= dt, \\ \zeta_2^{(1)} &= \sin \theta_2 d\theta_1 - d\theta_3, \\ \zeta_3^{(1)} &= \cos \theta_2 \cos \theta_3 d\theta_1 - \sin \theta_3 d\theta_2, \\ \zeta_4^{(1)} &= \cos \theta_2 \sin \theta_3 d\theta_1 + \cos \theta_3 d\theta_2, \\ \zeta_5^{(1)} &= \cos \theta_1 \cos \theta_2 dx'_1 - \sin \theta_1 \cos \theta_2 dx'_2 - \sin \theta_2 dx'_3, \\ \zeta_6^{(1)} &= (\cos \theta_1 \cos \theta_2 x'_1 - \sin \theta_1 \cos \theta_2 x'_2 - \sin \theta_2 x'_3) dt \end{aligned}$$



$$\begin{aligned}
& -\cos \theta_1 \cos \theta_2 dx_1 + \sin \theta_1 \cos \theta_2 dx_2, \\
\zeta_7^{(1)} &= -(\sin \theta_1 \sin \theta_3 + \cos \theta_1 \sin \theta_2 \cos \theta_3) dx_1' \\
& + (\sin \theta_1 \sin \theta_2 \cos \theta_3 - \cos \theta_1 \sin \theta_3) dx_2' - \cos \theta_2 \cos \theta_3 dx_3' \\
\zeta_8^{(1)} &= (\cos \theta_1 \sin \theta_2 \sin \theta_3 - \sin \theta_1 \cos \theta_3) dx_1', \\
& - (\sin \theta_1 \sin \theta_2 \sin \theta_3 + \cos \theta_1 \cos \theta_3) dx_2' + \cos \theta_2 \sin \theta_3 dx_3', \\
\zeta_9^{(1)} &= \left( (\sin \theta_1 \sin \theta_3 + \cos \theta_1 \sin \theta_2 \cos \theta_3) x_1' + (\cos \theta_1 \sin \theta_3 \right. \\
& \left. - \sin \theta_1 \sin \theta_2 \cos \theta_3) x_2' + \cos \theta_2 \cos \theta_3 x_3' \right) dt \\
& - (\cos \theta_1 \sin \theta_2 \cos \theta_3 + \sin \theta_1 \sin \theta_3) dx_1 \\
& + (-\cos \theta_1 \sin \theta_3 + \sin \theta_1 \sin \theta_2 \cos \theta_3) dx_2 - \cos \theta_2 \cos \theta_3 dx_3, \\
\zeta_{10}^{(1)} &= \left( (\sin \theta_1 \cos \theta_3 - \cos \theta_1 \sin \theta_2 \sin \theta_3) x_1' + (\cos \theta_1 \cos \theta_3 \right. \\
& \left. - \sin \theta_1 \sin \theta_2 \sin \theta_3) x_2' + \cos \theta_2 \sin \theta_3 x_3' \right) dt \\
& - (\cos \theta_1 \sin \theta_2 \cos \theta_3 + \sin \theta_1 \sin \theta_3) dx_1 \\
& + (\sin \theta_1 \sin \theta_2 \cos \theta_3 - \cos \theta_1 \sin \theta_3) dx_2 - \cos \theta_2 \cos \theta_3 dx_3,
\end{aligned}$$

which is an invariant coframe on  $\mathcal{B}_1 = \{(t, x, x_t, \theta)\} \cong J^1(\mathbb{R}; \mathbb{R}^3) \times G_3 \cong \mathbb{R}^7 \times \text{SO}(3)$   $\square$ .

## 5 Second order moving coframes

By restricting  $\zeta^{(1)}$  to the second prolongation  $J^2x \times G_3$ , which is a four dimensional manifold, we have

**Theorem 9.** *The restricted (or horizontal) moving coframe forms are explicitly given by*

$$\begin{aligned}
\eta_1^{(1)} &= dt, \\
\eta_2^{(1)} &= \sin \theta_2 d\theta_1 - d\theta_3, \\
\eta_3^{(1)} &= \cos \theta_2 \cos \theta_3 d\theta_1 - \sin \theta_3 d\theta_2, \\
\eta_4^{(1)} &= \cos \theta_2 \sin \theta_3 d\theta_1 + \cos \theta_3 d\theta_2,
\end{aligned}$$

and their linear dependencies are,  $\eta_5^{(1)} = J_1 \eta_1^{(1)}$ ,  $\eta_7^{(1)} = J_2 \eta_1^{(1)}$ ,  $\eta_8^{(1)} = J_3 \eta_1^{(1)}$ , and  $\eta_6^{(1)} = \eta_9^{(1)} = \eta_{10}^{(1)} = 0$ , where

$$\begin{aligned}
J_1 &= (\cos \theta_1 \sin \theta_2 \sin \theta_3 - \sin \theta_1 \cos \theta_3) x_1'' \\
& - (\sin \theta_1 \sin \theta_2 \sin \theta_3 + \cos \theta_1 \cos \theta_3) x_2'' + \cos \theta_2 \sin \theta_3 x_3'', \\
J_2 &= -(\sin \theta_1 \sin \theta_3 + \cos \theta_3 \sin \theta_2 \cos \theta_1) x_1'' \\
& + (\sin \theta_1 \sin \theta_2 \cos \theta_3 - \cos \theta_1 \sin \theta_3) x_2'' - \cos \theta_2 \cos \theta_3 x_3'',
\end{aligned}$$

$$J_3 = -\cos \theta_2 \cos \theta_1 x_1'' + \cos \theta_2 \sin \theta_1 x_2'' + \sin \theta_2 x_3''.$$

If we assume  $(J_1, J_2, J_3) = (-a, 0, 0)$ , where the length of acceleration  $\|\mathbf{x}_{tt}\|$  is denoted by  $a$ , then we have  $R\mathbf{x}_{tt} = (a, 0, 0)$ , and by simple computations, have

$$\theta_1 = -\arctan\left(\frac{x_2''}{x_1''}\right) \quad \text{and} \quad \theta_2 = \arcsin\left(\frac{x_3''}{\|\mathbf{x}_{tt}\|}\right).$$

It can be also easily seen that  $a$  is an invariant.

Now we choose a cross section  $K = \{t = 0, \mathbf{x} = 0, \mathbf{x}_t = 0, \|\mathbf{x}_{tt}\| = a, \theta = 0\}$ . By recomputing the forms  $\zeta^{(2)} = \rho^{(2)*}\mu$ , we have

**Theorem 10.** *The second order moving coframe is*

$$\begin{aligned} \zeta_1^{(2)} &= dt, \\ \zeta_2^{(2)} &= -d\theta_3 + \frac{x_2''x_3''dx_1''}{a(x_1''^2 + x_2''^2)} - \frac{x_1''x_3''dx_2''}{a(x_1''^2 + x_2''^2)}, \\ \zeta_3^{(2)} &= \frac{x_1''x_3''\sin\theta_3 + ax_2''\cos\theta_3}{a^2\sqrt{x_1''^2 + x_2''^2}}dx_1'' - \frac{x_2''x_3''\sin\theta_3 + ax_1''\cos\theta_3}{a^2\sqrt{x_1''^2 + x_2''^2}}dx_2'' \\ &\quad + \frac{\sin\theta_3\sqrt{x_1''^2 + x_2''^2}}{a^2}dx_3'', \\ \zeta_4^{(2)} &= -\frac{x_1''x_3''\cos\theta_3 + ax_2''\sin\theta_3}{a^2\sqrt{x_1''^2 + x_2''^2}}dx_1'' - \frac{x_2''x_3''\cos\theta_3 + ax_1''\sin\theta_3}{a^2\sqrt{x_1''^2 + x_2''^2}}dx_2'' \\ &\quad + \frac{\cos\theta_3\sqrt{x_1''^2 + x_2''^2}}{a^2}dx_3'', \\ \zeta_5^{(2)} &= -\frac{x_1''}{a}dx_1' - \frac{x_2''}{a}dx_2' - \frac{x_3''}{a}dx_3', \\ \zeta_6^{(2)} &= \frac{x_1'x_1'' + x_2'x_2'' + x_3'x_3''}{a}dt - \frac{x_1''}{a}dx_1 - \frac{x_2''}{a}dx_2 - \frac{x_3''}{a}dx_3, \\ \zeta_7^{(2)} &= \frac{ax_2''\sin\theta_3 - x_1''x_3''\cos\theta_3}{a\sqrt{x_1''^2 + x_2''^2}}dx_1 - \frac{ax_1''\sin\theta_3 + x_2''x_3''\cos\theta_3}{a\sqrt{x_1''^2 + x_2''^2}}dx_2 \\ &\quad - \frac{\sqrt{x_1''^2 + x_2''^2}\cos\theta_3}{a}dx_3, \\ \zeta_8^{(2)} &= \frac{ax_2''\cos\theta_3 - x_1''x_3''\sin\theta_3}{a\sqrt{x_1''^2 + x_2''^2}}dx_1 + \frac{-ax_1''\cos\theta_3 + x_2''x_3''\sin\theta_3}{a\sqrt{x_1''^2 + x_2''^2}}dx_2 \\ &\quad + \frac{\sqrt{x_1''^2 + x_2''^2}\sin\theta_3}{a}dx_3, \\ \zeta_9^{(2)} &= \frac{a(x_2'x_1'' - x_1'x_2'')\cos\theta_3 - (x_2'x_3''x_2'' + x_1'x_3''x_1'' + x_3'x_1''^2 + x_3'x_2''^2)\sin\theta_3}{a\sqrt{x_1''^2 + x_2''^2}}dt \\ &\quad + \frac{ax_2''\cos\theta_3 + x_1''x_3''\sin\theta_3}{a\sqrt{x_1''^2 + x_2''^2}}dx_1 - \frac{ax_1''\cos\theta_3 - x_2''x_3''\sin\theta_3}{a\sqrt{x_1''^2 + x_2''^2}}dx_2 \end{aligned}$$

$$\begin{aligned}
& + \frac{\sin \theta_3 \sqrt{x_1''^2 + x_2''^2}}{a} dx_3, \\
\zeta_{10}^{(2)} = & \frac{a(x_2'x_1'' - x_1'x_2'') \sin \theta_3 + (x_2'x_3''x_2'' + x_1'x_3''x_1'' + x_3'x_1''^2 + x_3'x_2''^2) \cos \theta_3}{a \sqrt{x_1''^2 + x_2''^2}} dt \\
& + \frac{ax_2'' \sin \theta_3 - x_1''x_3'' \cos \theta_3}{a \sqrt{x_1''^2 + x_2''^2}} dx_1 - \frac{ax_1'' \sin \theta_3 + x_2''x_3'' \cos \theta_3}{a \sqrt{x_1''^2 + x_2''^2}} dx_2 \\
& - \frac{\cos \theta_3 \sqrt{x_1''^2 + x_2''^2}}{a} dx_3.
\end{aligned}$$

For any constant  $a > 0$ , these forms serve as a coframe on

$$\begin{aligned}
\mathcal{B}_2 = & \left\{ (t, \mathbf{x}, \mathbf{x}_t, \mathbf{x}_{tt}, \theta) \in \mathbb{R}^{10} \times \text{SO}(3) \mid \right. \\
& \left. \theta_1 = -\arctan\left(\frac{x_2''}{x_1''}\right), \theta_2 = \arcsin\left(\frac{x_3''}{\|\mathbf{x}_{tt}\|}\right), \|\mathbf{x}_{tt}\| = a \right\}
\end{aligned}$$

## 6 Third order moving coframes

By restricting to the third prolongation  $J^3\mathcal{B}_2$  which is a 2-dimensional manifold, we have,

**Theorem 11.** *The restricted (or horizontal) moving coframe forms are explicitly given by  $\eta_1^{(2)} = dt$ ,*

$$\eta_2^{(2)} = \frac{x_3''(x_1'x_2'' - x_2'x_1'')}{a(x_1''^2 + x_2''^2)} \eta_1^{(2)} - d\theta_3,$$

their linear dependencies in this step are  $\eta_3^{(2)} = J_1\eta_1^{(2)}$ ,  $\eta_4^{(2)} = J_2\eta_1^{(2)}$ ,  $\eta_5^{(2)} = -a\eta_1^{(2)}$  and  $\eta_6^{(2)} = \eta_7^{(2)} = \eta_8^{(2)} = \eta_9^{(2)} = \eta_{10}^{(2)} = 0$ ; Where

$$\begin{aligned}
J_1 = & \frac{1}{a^2 \sqrt{x_1''^2 + x_2''^2}} \left\{ a(x_1^{(3)}x_2'' - x_2^{(3)}x_1'') \cos \theta_3 \right. \\
& \left. + ((x_1''x_1^{(3)} + x_2^{(3)}x_2'')x_3'' - (x_1''^2 + x_2''^2)x_3^{(3)}) \sin \theta_3 \right\}, \\
J_2 = & \frac{1}{a^2 \sqrt{x_1''^2 + x_2''^2}} \left\{ a(x_1^{(3)}x_2'' - x_2^{(3)}x_1'') \sin \theta_3 \right. \\
& \left. + ((x_1''^2 + x_2''^2)x_3^{(3)} - (x_1''x_1^{(3)} + x_2^{(3)}x_2'')x_3'') \cos \theta_3 \right\}.
\end{aligned}$$

If we assume  $J_1 = 0$ , then we find that  $J_2 = \frac{\|\mathbf{x}_{tt} \times \mathbf{x}_{ttt}\|}{a^2}$  and

$$\theta_3 = \arctan \left( \frac{a(x_1^{(3)}x_2'' - x_2^{(3)}x_1'')}{(x_1''^2 + x_2''^2)x_3^{(3)} - (x_1''x_1^{(3)} + x_2^{(3)}x_2'')x_3''} \right).$$

thus,

**Theorem 12.** *The most general third order compatible lift has the form*

$$\rho^{(3)}(t, \mathbf{x}; \mathbf{x}_t, \mathbf{x}_{tt}, \mathbf{x}_{ttt}) = \begin{bmatrix} 1 & 0 & 0 & 0 & t \\ \mathbf{x}_t & \frac{\mathbf{x}_{tt}}{\|\mathbf{x}_{tt}\|} & \frac{\mathbf{x}_{tt} \times \mathbf{x}_{ttt}}{\|\mathbf{x}_{tt} \times \mathbf{x}_{ttt}\|} & \frac{\mathbf{x}_{tt} \times (\mathbf{x}_{ttt} \times \mathbf{x}_{ttt})}{\|\mathbf{x}_{tt} \times (\mathbf{x}_{ttt} \times \mathbf{x}_{ttt})\|} & \mathbf{x} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

**Theorem 13.** *The third order moving coframe  $\zeta^{(3)} = \rho^{(3)*}\mu$  is*

$$\begin{aligned} \zeta_1^{(3)} &= dt, \\ \zeta_2^{(3)} &= \frac{\mathbf{x}_{tt} \times \mathbf{x}_{ttt}}{a\|\mathbf{x}_{tt} \times \mathbf{x}_{ttt}\|^2} \cdot \left( (\mathbf{x}_{tt} \cdot \mathbf{x}_{ttt}) d\mathbf{x}_{tt} + a^2 d\mathbf{x}_{ttt} \right), \\ \zeta_3^{(3)} &= -\frac{\mathbf{x}_{tt} \cdot \mathbf{x}_{ttt}}{a\|\mathbf{x}_{tt} \times \mathbf{x}_{ttt}\|} (\mathbf{x}_{tt} \times \mathbf{x}_{ttt}) \cdot d\mathbf{x}_{tt}, \\ \zeta_4^{(3)} &= -\frac{\mathbf{x}_{tt} \times (\mathbf{x}_{tt} \times \mathbf{x}_{ttt})}{a^2\|\mathbf{x}_{tt} \times \mathbf{x}_{ttt}\|} \cdot d\mathbf{x}_{tt}, \\ \zeta_5^{(3)} &= -\frac{1}{a}\mathbf{x}_{tt} \cdot d\mathbf{x}_t, \\ \zeta_6^{(3)} &= \frac{\mathbf{x}_t \cdot \mathbf{x}_{tt}}{a} dt - \frac{1}{a}\mathbf{x}_{tt} \cdot d\mathbf{x}, \\ \zeta_7^{(3)} &= \frac{\mathbf{x}_{tt} \times (\mathbf{x}_{tt} \times \mathbf{x}_{ttt})}{a\|\mathbf{x}_{tt} \times \mathbf{x}_{ttt}\|} \cdot d\mathbf{x}_t, \\ \zeta_8^{(3)} &= \frac{\mathbf{x}_{tt} \times \mathbf{x}_{ttt}}{\|\mathbf{x}_{tt} \times \mathbf{x}_{ttt}\|} \cdot d\mathbf{x}_t, \\ \zeta_9^{(3)} &= -\frac{\mathbf{x}_t \cdot (\mathbf{x}_{tt} \times \mathbf{x}_{ttt})}{\|\mathbf{x}_{tt} \times \mathbf{x}_{ttt}\|} dt + \frac{\mathbf{x}_{tt} \times \mathbf{x}_{ttt}}{\|\mathbf{x}_{tt} \times \mathbf{x}_{ttt}\|} \cdot d\mathbf{x}, \\ \zeta_{10}^{(3)} &= \frac{(\mathbf{x}_t \times \mathbf{x}_{tt}) \cdot (\mathbf{x}_{tt} \times \mathbf{x}_{ttt})}{a\|\mathbf{x}_{tt} \times \mathbf{x}_{ttt}\|} dt - \frac{\mathbf{x}_{tt} \times (\mathbf{x}_{tt} \times \mathbf{x}_{ttt})}{a\|\mathbf{x}_{tt} \times \mathbf{x}_{ttt}\|} \cdot d\mathbf{x}. \end{aligned}$$

For any constant  $a > 0$ , these forms serve as a coframe on

$$\begin{aligned} \mathcal{B}_3 &= \left\{ (t, \mathbf{x}, \mathbf{x}_t, \mathbf{x}_{tt}, \theta) \in \mathbb{R}^{10} \times \text{SO}(3) \mid \right. \\ &\quad \theta_1 = -\arctan\left(\frac{x_2''}{x_1''}\right), \theta_2 = \arcsin\left(\frac{x_3''}{\|\mathbf{x}_{tt}\|}\right), \|\mathbf{x}_{tt}\| = a, \\ &\quad \left. \theta_3 = \arctan\left(\frac{a(x_1^{(3)}x_2'' - x_2^{(3)}x_1'')}{(x_1''^2 + x_2''^2)x_3^{(3)} - (x_1''x_1^{(3)} + x_2''x_2^{(3)})x_3''}\right) \right\}. \end{aligned}$$

**Theorem 14.** *The restricted (or horizontal) moving coframe forms are explicitly given by  $\eta_1^{(3)} = dt$ ,  $\eta_4^{(3)} = \frac{\|\mathbf{x}_{tt} \times \mathbf{x}_{ttt}\|}{a^2}\eta_1^{(3)}$ ,  $\eta_3^{(3)} = \eta_5^{(3)} = \eta_6^{(3)} = \eta_7^{(3)} = \eta_9^{(3)} = \eta_{10}^{(3)} = 0$ ,  $\eta_8^{(3)} = -a\eta_1^{(3)}$ , and  $\eta_2^{(3)} = J\eta_1^{(3)}$ , where  $J = a((\mathbf{x}_{tt} \times \mathbf{x}_{ttt}) \cdot \mathbf{x}_{ttt})/\|\mathbf{x}_{tt} \times \mathbf{x}_{ttt}\|^2$ .*

**Theorem 15.**  $\frac{d}{dt}$  is a differential operator and the functions  $a_1 = \|\mathbf{x}_{tt}\|$ ,  $a_2 = \|\mathbf{x}_{tt} \times \mathbf{x}_{ttt}\|$  and  $a_3 = (\mathbf{x}_{tt} \times \mathbf{x}_{ttt}) \cdot \mathbf{x}_{tttt}$  are differential invariants.

## 7 Dimensional considerations

In this section, we use the conventions of chapter 5 of [5].

If we use the coordinates  $(t, \mathbf{x}, \mathbf{x}_t, \mathbf{x}_{tt}, \dots, \mathbf{x}^{(n)})$  for  $J^n(\mathbb{R}; \mathbb{R}^3)$ , then the prolonged group action  $\text{SGal}(n)$  on  $J^n(\mathbb{R}; \mathbb{R}^3)$  can be written as  $\bar{t} = t + s$ ,  $\bar{\mathbf{x}} = R\mathbf{x} + t\mathbf{v} + \mathbf{y}$ ,  $\bar{\mathbf{x}}_t = R\mathbf{x}_t + \mathbf{v}$ , and  $\bar{\mathbf{x}}^{(n)} = R\mathbf{x}^{(n)}$  for  $n \geq 2$ .

It is recommended that the dimension of  $J^n(\mathbb{R}; \mathbb{R}^3)$  is  $p + q^n = 3n + 4$ , and the dimension of  $\text{SGal}^{(n)}$  is 10.

**Theorem 16.** *The following functions are differential invariants:*

- 1)  $I_n = \|\mathbf{x}^{(n)}\|$  for  $n \geq 2$ .
- 2)  $J_{n,m} = \mathbf{x}^{(n)} \cdot \mathbf{x}^{(m)}$  for  $n > m \geq 2$ .
- 3)  $K_{n,m} = \|\mathbf{x}^{(n)} \times \mathbf{x}^{(m)}\|$  for  $n > m \geq 2$ .
- 4)  $L_{l,n,m} = (\mathbf{x}^{(l)} \times \mathbf{x}^{(n)}) \cdot \mathbf{x}^{(m)}$  for  $l > n > m \geq 2$ .

*Proof:* If  $n, m \geq 2$ , then since  $\bar{\mathbf{x}}^{(n)} = R\mathbf{x}^{(n)}$ ,  $\bar{\mathbf{x}}^{(m)} = R\mathbf{x}^{(m)}$  and  $R \in \text{SO}(3)$ , therefore  $\bar{\mathbf{x}}^{(n)} \cdot \bar{\mathbf{x}}^{(m)} = \mathbf{x}^{(n)} \cdot \mathbf{x}^{(m)}$ ; hence  $I_{n,m}$  is an invariant.

By (1), (2) and formulas  $\|u \times v\|^2 = \|u\|^2\|v\|^2 - (u \cdot v)^2$ , we find that  $K_{n,m} = I_n I_m - J_{n,m}^2$  is an invariant.

Since  $(u_1 \times u_2) \cdot u_3 = \det(u_i \cdot u_j)$ , then  $L_{l,n,m}$  is a function of  $J_{n,m}$ 's, and this complete the proof.  $\square$

By usual computations, we find that the maximal dimension of prolonged action are:  $s_0 = 4$ ,  $s_1 = 7$ ,  $s_2 = 9$ ,  $s_n = 10$  for  $n \geq 3$ . Therefore, the order of this group action is  $s = 3$ .

Therefore, the  $i_n$  functionally independent differential invariants of order at most  $n$  are:  $i_0 = i_1 = 0$ ,  $i_2 = 1$  and  $i_n = 3(n - 2)$  for  $n \geq 3$ .

By the theorem 5.31 of [5], we have

**Theorem 17.** *The complete system of 3<sup>rd</sup> order differential invariants of special Galilean group action are  $\|\mathbf{x}_{tt}\|$ ,  $\|\mathbf{x}_{ttt}\|$  and  $\mathbf{x}_{tt} \cdot \mathbf{x}_{ttt}$ . Locally, every 3<sup>rd</sup> order differential invariant of  $\text{SGal}(3)$  can be written as a function of these differential invariants.*  $\square$

**Corollary 1.**  $a_1 = I_2$ ,  $a_2 = J_{3,2}$ ,  $a_3 = L_{4,3,2}/K_{3,2}^2$ .  $\square$

**Theorem 18.** *Every differential invariant of special Galilean group action is a function of  $a = \|\mathbf{x}_{tt}\|$ ,  $b = \|\mathbf{x}_{ttt}\|$  and their derivatives with respect to  $t$ .*

*Proof:* According to theorem 17, it is enough to show that  $\mathbf{x}_{tt} \cdot \mathbf{x}_{ttt}$  can be written as a function of  $\|\mathbf{x}_{tt}\|$  and  $\|\mathbf{x}_{ttt}\|$ . But  $\frac{1}{2} \frac{d}{dt} \|\mathbf{x}_{tt}\|^2 = \mathbf{x}_{tt} \cdot \mathbf{x}_{ttt}$ .  $\square$

## 8 $\{e\}$ -structure

The necessary condition for local special Galilean equivalence of two given motions is that the corresponding invariants are the same. These produce a large amount of necessary conditions.

For sufficient condition of equivalence for coframes  $\zeta$  we can rewrite two-forms  $d\zeta_i$  in terms of wedge products of the  $\zeta_i$ 's. This produces the *structure functions*. There are our *original invariants*, by differentiation from them we have *derived invariants*, now we can construct a large collection of invariants, whose functional interrelationships provide a necessary condition for equivalence. It is time to continue by introducing *structure invariants*. This latter structure serves to define the components of the *structure map*. The  $s^{th}$  order *classifying space* and the fully regularity condition on  $s^{th}$  order *structure map* leads us to the definition of  $s^{th}$  order *classifying manifold*  $\mathcal{C}^{(s)}$  due to chapter 8 in [5]. In view of the proposition 8.11 in [5], necessary conditions for the (local) equivalence of coframes are that for each  $s \geq 0$ , their  $s^{th}$  order classifying manifolds are overlap. Now the fully regularity conditions provide that these necessary conditions are also sufficient.

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